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Research Article

A Class of p-q-Laplacian Type Equation with Potentials Eigenvalue Problem in $\mathbb{R}^N$

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The nonlinear elliptic eigenvalue problem $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \mathrm{div}(|\nabla u|^{q-2}\nabla u) + \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u = f(x,u), u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$, where $2 \leq q \leq p < N$ and $a(x) \in L^{N/p}(\mathbf{R}^N), b(x) \in L^{N/q}(\mathbf{R}^N), a(x), b(x) > 0$ are studied. The key ingredient is a special constrained minimization method.

1. Introduction

In this paper, we are interested in finding nontrivial weak solutions for the nonlinear eigenvalue problem

$$\begin{gathered} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) - \mathrm{div}\left(|\nabla u|^{q-2}\nabla u\right) + a(x)|u|^{p-2}u + b(x)|u|^{q-2}u = f(x,u), \\ u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N), \quad u \neq 0, \end{gathered} \tag{1.1}$$

where $2 \leq q \leq p < N$ and $a(x) \in L^{N/p}(\mathbf{R}^N)$, $b(x) \in L^{N/q}(\mathbf{R}^N)$, $a(x), b(x) > 0$, $\inf a(x), \inf b(x) \neq 0$, $f(x,u)$ satisfy the following conditions:

(A) $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$, $\lim_{t\to 0}(f(x,t)/|t|^{p-1}) = 0$, and $\lim_{|t|\to\infty}(f(x,t)/|t|^{p-1+p^2/N}) = 0$ uniformly in $x \in \mathbf{R}^N$,

(B) $\lim_{|x|\to\infty} f(x,t) = \overline{f}(t)$ uniformly for t in bounded subsets of R.

Remark 1.1. We can see if $a(x) \in L^{N/p}(\mathbf{R}^N)$, $b(x) \in L^{N/q}(\mathbf{R}^N)$, then

$$\int_{\mathbf{R}^N} a(x)|u|^p dx < \left(\int_{\mathbf{R}^N} a(x)^{N/p}\right)^{1-p/p^*}\left(\int_{\mathbf{R}^N} u^{p^*}\right)^{p/p^*} < \infty,$$
$$\int_{\mathbf{R}^N} b(x)|u|^q dx < \left(\int_{\mathbf{R}^N} b(x)^{N/q}\right)^{1-q/q^*}\left(\int_{\mathbf{R}^N} u^{q^*}\right)^{q/q^*} < \infty, \tag{1.2}$$

where $p^* = Np/(N-p)$ and $q^* = Nq/(N-q)$.

Problem (1.1) comes, for example, from a general reaction-diffusion system:

$$u_t = \operatorname{div}[D(u)\nabla u] + c(x,u), \tag{1.3}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.3) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ is a polynomial of u with variable coefficients.

When $p = q = 2$, problem (1.1) is a normal Schrodinger equation which has been extensively studied, for example, [1–8]. The authors used many different methods to study the equation. In [8], the authors established some embedding results of weighted Sobolev spaces of radially symmetric functions which are used to obtain ground state solutions. In [6], the authors studied the equation depending upon the local behavior of V near its global minimum. In [3], the authors used spectral properties of the Schrodinger operator to study nonlinear Schrodinger equations with steep potential well. In [9], the author imposed on functions k and K conditions ensuring that this problem can be written in a variational form. We know that $W^{1,p}(\mathbf{R}^N)$ is not a Hilbert space for $1 < p < N$, except for $p = 2$. The space $W^{1,p}(\mathbf{R}^N)$ with $p \neq 2$ does not satisfy the Lieb lemma (e.g., see [9]). And $\mathbf{R}^N$ results in the loss of compactness. So there are many difficulties to study equation (1.1) of $p = q \neq 2$ by the usual methods. There seems to be little work on the case $p = q \neq 2$ for problem (1.1), to the best of our knowledge. In this paper, we overcome these difficulties and study (1.1) of $p \geq q \geq 2$.

Recently, when $p = q$, $a(x) = b(x)$, and $f(x,u) = 0$ then the problem is the following eigenvalue problem has been studied by many authors:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = V(x)|u|^{p-2}u,$$
$$u \in D_0^{1,p}(\Omega), \quad u \neq 0, \tag{1.4}$$

where $\Omega \subseteq \mathbf{R}^N$. We can see [10–13]. In [13], Szulkin and Willem generalized several earlier results concerning the existence of an infinite sequence of eigenvalues.

When $p = q$ and $a(x), b(x)$ is constant then the problem is the following quasilinear elliptic equation:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda|u|^{p-2}u = f(x,u), \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \quad u \neq 0, \tag{1.5}$$

where $1 < p < N$, $N \geq 3$, λ is a parameter, Ω is an unbounded domain in $\mathbf{R}^N$. There are many results about it we can see [14–18]. Because of the unboundedness of the domain, the Sobolev compact embedding does not hold. There are many methods to overcome the difficulty. In [15], the authors used the concentration-compactness principle posed by P. L. Lions and the mountain pass lemma to solve problem (1.5). In [17, 18], the authors studied the problem in symmetric Sobolev spaces which possess Sobolev compact embedding. By the result and a min-max procedure formulated by Bahri and Li [16], they considered the existence of positive solutions of

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^{p-1} = q(x)u^{\alpha} \quad \text{in } \mathbf{R}^N, \tag{1.6}$$

where $q(x)$ satisfies some conditions. We can see if λ is function, then it cannot easily be proved by the above methods.

When $a(x), b(x)$ is positive constant, He and Li used the mountain pass theorem and concentration-compactness principle to study the following elliptic problem in [19]:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \operatorname{div}\left(|\nabla u|^{q-2}\nabla u\right) + m|u|^{p-2}u + n|u|^{q-2}u = f(x,u) \quad \text{in } \mathbf{R}^N, \\ u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N), \tag{1.7}$$

where $m, n > 0$, $N \geq 3$, and $1 < q < p < N$, $f(x,u)/u^{p-1}$ tends to a positive constant l as $u \to +\infty$. The authors prove in this paper that the problem possesses a nontrivial solution even if the nonlinearity $f(x,t)$ does not satisfy the Ambrosetti-Rabinowitz condition.

In [20], Li and Liang used the mountain pass theorem to study the following elliptic problem:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \operatorname{div}\left(|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + |u|^{q-2}u = f(x,u) \quad \text{in } \mathbf{R}^N, \\ u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N), \tag{1.8}$$

where $1 < q < p < N$. They generalized a similar result for p-Laplacian type equation in [15].

It is our purpose in this paper to study the existence of ground state to the problem (1.1) in $\mathbf{R}^N$. We call any minimizer a ground state for (1.1). We inspired by [9, 16, 21] try to use constrained minimization method to study problem (1.1). Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all. But since both p- and q-Laplacian operators are involved, careful analysis is needed. A typical difficulty for problem (1.1) in $\mathbf{R}^N$ is the lack of compactness of the Sobolev imbedding due to the invariance of $\mathbf{R}^N$ under the translations and rotations. However, our method has essential difference with the methods used in [19, 20]. In order to obtain the results, we have to overcome two main difficulties; one is that $\mathbf{R}^N$ results in the loss of compactness; the other is that $W^{1,p}(\mathbf{R}^N)$ is not a Hilbert space for $1 < p < N$ and it does not satisfy the Lieb lemma, except for $p = 2$.

The paper is organized as follows. In Section 2, we state some condition and many lemmas which we need in the proof of the main theorem. In Section 3, we give the proof of the main result of the paper.

2. Preliminaries

Let

$$F(x,t) = \int_0^t f(x,s)ds, \qquad \overline{F}(t) = \int_0^t \overline{f}(s)ds \tag{2.1}$$

and we define variational functionals $I : W^{1,p} \cap W^{1,q}(\mathbf{R}^N) \to R$ and $I^\infty : W^{1,p} \cap W^{1,q}(\mathbf{R}^N) \to R$ by

$$\begin{aligned} I(u) &= \frac{1}{p}\int_{\mathbf{R}^N} |\nabla u|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla u|^q dx - \int_{\mathbf{R}^N} F(x,u)dx, \\ I^\infty(u) &= \frac{1}{p}\int_{\mathbf{R}^N} |\nabla u|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla u|^q dx - \int_{\mathbf{R}^N} \overline{F}(u)dx. \end{aligned} \tag{2.2}$$

Solutions to problem (1.1) will be found as minimizers of the variational problem

$$I_\lambda = \inf\left\{ I(u); u \in W^{1,p}\left(\mathbf{R}^N\right), \int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx = \lambda \right\}, \quad \lambda > 0. \tag{I_λ}$$

To find a solution of problem (I_λ) we introduce the (limit) variational problem

$$I_\lambda^\infty = \inf\left\{ I^\infty(u); u \in W^{1,p}\left(\mathbf{R}^N\right), \int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx = \lambda \right\}, \quad \lambda > 0. \tag{I_λ^∞}$$

Lemma 2.1. *Let $(u_n) \subseteq W_0^{1,p}(\Omega)$ a bounded sequence and $p \geq 2$. Going if necessary to a subsequence, one may assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_n \to u$ almost everywhere, where $\Omega \subseteq \mathbf{R}^N$ is an open subset.*

Then,

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^p dx \geq \lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^p dx + \lim_{n\to\infty}\int_\Omega |\nabla u|^p dx. \tag{2.3}$$

Proof. When $p = 2$ from Brezis-Lieb lemma (see [21, Lemma 1.32]) we have

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2 dx = \lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^2 dx + \lim_{n\to\infty}\int_\Omega |\nabla u|^2 dx, \tag{2.4}$$

when $3 \geq p > 2$, using the lower semicontinuity of the L^p-norm with respect to the weak convergence and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, we deduce

$$\begin{aligned}
\left\langle |\nabla u_n|^{p-2}\nabla u_n, \nabla u_n \right\rangle &\geq \left\langle |\nabla u|^{p-2}\nabla u, \nabla u \right\rangle + o(1), \\
\lim_{n\to\infty}\left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u_n, \nabla u_n \right\rangle &\geq \lim_{n\to\infty}\left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u_n, \nabla u \right\rangle \\
&= \lim_{n\to\infty}\left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u, \nabla u_n \right\rangle \\
&= \lim_{n\to\infty}\left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u, \nabla u \right\rangle.
\end{aligned} \tag{2.5}$$

Then,

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^p - |\nabla u|^p\right) dx \\
&\quad = \lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p-2}\left(|\nabla u_n|^2 - |\nabla u|^2\right) dx + \lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2} - |\nabla u|^{p-2}\right)|\nabla u|^2 dx \\
&\quad = \lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2} + |\nabla u|^{p-2}\right)\left(|\nabla u_n|^2 - |\nabla u|^2\right) dx \\
&\qquad + \lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2}|\nabla u|^2 - |\nabla u|^{p-2}|\nabla u_n|^2\right) dx.
\end{aligned} \tag{2.6}$$

From $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$,

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2}|\nabla u|^2 - |\nabla u|^{p-2}|\nabla u_n|^2\right) dx = 0. \tag{2.7}$$

So

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^p - |\nabla u|^p\right) dx &= \lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2} + |\nabla u|^{p-2}\right)\left(|\nabla u_n|^2 - |\nabla u|^2\right) dx \\
&\geq \lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^{p-2}\left(|\nabla u_n|^2 - |\nabla u|^2\right).
\end{aligned} \tag{2.8}$$

So we have

$$\left\langle |\nabla u_n|^{p-2}\nabla u_n, \nabla u_n \right\rangle + \left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u, \nabla u_n \right\rangle + \left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u_n, \nabla u \right\rangle \\ \geq \left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u_n, \nabla u_n \right\rangle + \left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u, \nabla u \right\rangle + \left\langle |\nabla u|^{p-2}\nabla u, \nabla u \right\rangle + o(1). \tag{2.9}$$

Then,

$$\left\langle |\nabla u_n|^{p-2}\nabla u_n, \nabla u_n \right\rangle \geq \left\langle |\nabla u_n - \nabla u|^{p-2}\nabla u_n - \nabla u, \nabla u_n - \nabla u \right\rangle + \left\langle |\nabla u|^{p-2}\nabla u, \nabla u \right\rangle + o(1) \\ \lim_{n\to\infty}\int_\Omega |\nabla u_n|^p dx \geq \lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^p dx + \lim_{n\to\infty}\int_\Omega |\nabla u|^p dx, \tag{2.10}$$

when $p > 3$, there exists a $k \in N$ that $0 < p - k \leq 1$. Then, we only need to prove the following inequality:

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^p - |\nabla u|^p\right)dx \geq \lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^{p-k}\left(|\nabla u_n|^k - |\nabla u|^k\right). \tag{2.11}$$

The proof of it is similar to the above, so we omit it here. So, the lemma is proved. □

Lemma 2.2. *Let $\{u_n\}$ be a bounded sequence in $W^{1,p}(\mathbf{R}^N)$ such that*

$$\lim_{n\to\infty}\sup_{y\in\mathbf{R}^N}\int_{B(y,R)} u_n^q dx = 0, \quad p \leq q < p^* \tag{2.12}$$

for some $R > 0$. Then $u_n \to 0$ in $L^s(\mathbf{R}^N)$ for $p < s < p^*$, where $p^* = Np/(N-p)$.

Proof. We consider the case $N \geq 3$. Let $q < s < p^*$ and $u \in W^{1,p}(\mathbf{R}^N)$. Holder and Sobolev inequalities imply that

$$|u|_{L^s(B(y,R))} \leq |u|^{1-\lambda}_{L^q(B(y,R))}|u|^{\lambda}_{L^{p^*}(B(y,R))} \\ \leq C|u|^{1-\lambda}_{L^q(B(y,R))}\left[\int_{(B(y,R))}\left(|u|^p + |\nabla u|^p\right)\right]^{\lambda/p}, \tag{2.13}$$

where $\lambda = ((s-q)/(p^*-q))(p^*/s)$. Choosing $\lambda = p/s$, we obtain

$$\int_{B(y,R)} |u|^s \leq C^s|u|^{(1-\lambda)s}_{L^q(B(y,R))}\int_{B(y,R)}\left(|u|^p + |\nabla u|^p\right). \tag{2.14}$$

Now, covering $\mathbf{R}^N$ by balls of radius r, in such a way that each point of $\mathbf{R}^N$ is contained in at most $N+1$ balls, we find

$$\int_{\mathbf{R}^N} |u|^s \leq (N+1)C^s \sup_{y\in\mathbf{R}^N} \left[\int_{B(y,R)} |u|^q\right]^{(1-\lambda)s/q} \int_{B(y,R)} \left(|u|^p + |\nabla u|^p\right). \tag{2.15}$$

Under the assumption of the lemma, $u_n \to 0$ in $L^s(\mathbf{R}^N)$, $p < s < p^*$. The proof is complete. □

Corollary 2.3. *Let $\{u_m\}$ be a sequence in $W^{1,p}(\mathbf{R}^N)$ satisfying $0 < \rho = \int_{\mathbf{R}^N} |u_m|^p dx$ and such that $u_m \rightharpoonup 0$ in $W^{1,p}(\mathbf{R}^N)$. Then there exist a sequence $\{y_m\} \subset \mathbf{R}^N$ and a function $0 \neq u \in W^{1,p}(\mathbf{R}^N)$ such that up to a subsequence $u_m(\cdot + y_m) \rightharpoonup u$ in $W^{1,p}(\mathbf{R}^N)$.*

Lemma 2.4. *Let $f \in C(\mathbf{R}^N \times \mathbf{R})$ and suppose that*

$$\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p^*-1}} = 0 \tag{2.16}$$

uniformly in $x \in \mathbf{R}^N$ and

$$\left|f(x,s)\right| \leq C\left(|s|^{p-1} + |s|^{p^*-1}\right) \tag{2.17}$$

for all $x \in \mathbf{R}^N$ and $t \in R$. If $u_m \rightharpoonup u_0$ in $W^{1,p}(\mathbf{R}^N)$ and $u_m \to u_0$ a.e. on $\mathbf{R}^N$, then

$$\lim_{m\to\infty} \left[\int_{\mathbf{R}^N} F(x,u_m)dx - \int_{\mathbf{R}^N} F(x,u_0)dx - \int_{\mathbf{R}^N} F(x,u_m-u_0)dx\right] = 0, \tag{2.18}$$

where $F(x,u) = \int_0^u f(x,t)dt$.

Proof. Let $R > 0$. Applying the mean value theorem we have

$$\begin{aligned}\int_{\mathbf{R}^N} F(x,u_m)dx &= \int_{|x|\leq R} F(x,u_m)dx + \int_{|x|\geq R} F(x,u_0 + (u_m - u_0))dx \\ &= \int_{|x|\leq R} F(x,u_m)dx + \int_{|x|\geq R} \left(F(x,u_m-u_0) + f(x,\theta u_0 + (u_m-u_0))u_0\right)dx,\end{aligned} \tag{2.19}$$

where θ depends on x and R and satisfies $0 < \theta < 1$. We now write

$$\begin{aligned}
&\left|\int_{\mathbf{R}^N} F(x,u_m)dx - \int_{\mathbf{R}^N} F(x,u_0)dx - \int_{\mathbf{R}^N} F(x,u_m-u_0)dx\right| \\
&\quad \le \left|\int_{|x|\le R} (F(x,u_m)-F(x,u_0))dx\right| + \left|\int_{|x|\ge R} F(x,u_0)dx\right| \\
&\qquad + \left|\int_{|x|\le R} F(x,u_m-u_0)dx\right| + \left|\int_{|x|\ge R} f(x,\theta u_0+(u_m-u_0)u_0)dx\right|.
\end{aligned} \tag{2.20}$$

For each fixed $R > 0$

$$\begin{gathered}
\lim_{m\to\infty}\int_{|x|\le R} (F(x,u_m)-F(x,u_0))dx = 0, \\
\lim_{m\to\infty}\int_{|x|\le R} F(x,u_m-u_0)dx = 0.
\end{gathered} \tag{2.21}$$

Applying (2.20) and the Holder inequality we get that

$$\begin{aligned}
&\left|\int_{|x|\ge R} f(x,\theta u_0+(u_m-u_0)u_0)dx\right| \\
&\quad \le C\int_{|x|\ge R}\Big(|\theta u_0+(u_m-u_0)|^{p-1}|u_0| + |\theta u_0+(u_m-u_0)|^{p^*-1}|u_0|\Big)dx \\
&\quad \le C\left(\int_{|x|\ge R}|u_0|^p\right)^{1/p}\left(\int_{|x|\ge R}|\theta u_0+(u_m-u_0)|^p\right)^{(p-1)/p} \\
&\qquad + C\left(\int_{|x|\ge R}|u_0|^{p^*}\right)^{1/p^*}\left(\int_{|x|\ge R}|\theta u_0+(u_m-u_0)|^{p^*}\right)^{(p^*-1)/p^*}.
\end{aligned} \tag{2.22}$$

Since $\{u_m\}$ is bounded in $W^{1,p}(\mathbf{R}^N)$ we see that

$$\lim_{R\to\infty}\left|\int_{|x|\ge R} f(x,\theta u_0+(u_m-u_0)u_0)dx\right| = 0. \tag{2.23}$$

The result follows from (2.21) and (2.23). □

Lemma 2.5. *Functions I_λ and I_λ^∞ are continuous on $(0,\infty)$ and minimizing sequences for problems (I_λ) and (I_λ^∞) are bounded in $W^{1,p}(\mathbf{R}^N)$.*

Proof. From condition (A), we observe that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|\overline{F}(u)\right|, |F(x,u)| \leq \varepsilon \int_{\mathbf{R}^N} |u|^p dx + \varepsilon \int_{\mathbf{R}^N} |u|^{p+p^2/N} dx + C_\varepsilon \int_{\mathbf{R}^N} |u|^\alpha dx, \tag{2.24}$$

where $p < \alpha < p + p^2/N$ and $\varepsilon > 0$.

By the Holder and Sobolev inequalities we have

$$\begin{aligned} \int_{\mathbf{R}^N} |u|^{p+p^2/N} dx &= \int_{\mathbf{R}^N} |u|^{p(p^*-p-p^2/N)/(p^*-p)+p^*(p^2/N)/(p^*-p)} dx \\ &\leq \left(\int_{\mathbf{R}^N} |u|^p\right)^{(p^*-p-p^2/N)/(p^*-p)} \left(\int_{\mathbf{R}^N} |u|^{p^*}\right)^{p^2/N/(p^*-p)} \\ &\leq S^{-1}\left(\int_{\mathbf{R}^N} |u|^p\right)^{p/N} \int_{\mathbf{R}^N} |\nabla u|^p dx, \end{aligned} \tag{2.25}$$

where $|u|_{p^*}^p \leq S^{-1}|\nabla u|_p^p$.

Similarly we have

$$\begin{aligned} \int_{\mathbf{R}^N} |u|^\alpha dx &= \int_{\mathbf{R}^N} |u|^{p((p^*-\alpha)/(p^*-p))+p^*((\alpha-p)/(p^*-p))} dx \\ &\leq \left(\int_{\mathbf{R}^N} |u|^p dx\right)^{(p^*-\alpha)/(p^*-p)} \left(\int_{\mathbf{R}^N} |u|^{p^*} dx\right)^{(\alpha-p)/(p^*-p)} \\ &\leq S^{-p^*(\alpha-p)/p(p^*-p)}\left(\int_{\mathbf{R}^N} |u|^p dx\right)^{(p^*-\alpha)/(p^*-p)} \left(\int_{\mathbf{R}^N} |\nabla u|^p dx\right)^{p^*(\alpha-p)/p(p^*-p)}. \end{aligned} \tag{2.26}$$

Consequently by the Young inequality we have

$$\int_{\mathbf{R}^N} |u|^\alpha dx \leq \eta \int_{\mathbf{R}^N} |\nabla u|^p dx + K(\eta)\left(\int_{\mathbf{R}^N} |u|^\alpha dx\right)^{p(p^*-\alpha)/(p^2p^*-p^2-p^*\alpha)} \tag{2.27}$$

for $\eta > 0$, where $K(\eta) > 0$ is a constant.

Because $u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$ so we can by Sobolev embedding and $\lambda = \int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx$ letting $\widehat{\lambda} = \int_{\mathbf{R}^N} |u|^p dx < \infty$, we derive the following estimates for $I(u)$ and $I^\infty(u)$:

$$\begin{aligned} I(u), I^\infty(u) \geq{} & \left(\frac{1}{p} - \varepsilon S^{-1}\widehat{\lambda}^{p/N} - C_\varepsilon \eta\right)\int_{\mathbf{R}^N} |\nabla u|^p dx \\ & + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla u|^q dx - \varepsilon\widehat{\lambda} - K(\eta)C_\varepsilon \widehat{\lambda}^{p(p^*-\alpha)/(p^2p^*-p^2-p^*\alpha)}. \end{aligned} \tag{2.28}$$

Choosing $\varepsilon > 0$ and $\eta > 0$ so that

$$\frac{1}{p} - \varepsilon S^{-1}\hat{\lambda}^{p/N} - C_\varepsilon \eta > 0, \tag{2.29}$$

we see that I_λ and I_λ^∞ are finite and moreover minimizing sequences for problems (I_λ) and (I_λ^∞) are bounded. It is easy to check that I_λ and I_λ^∞ are continuous on $(0,\infty)$. □

We observe that $I_\mu^\infty \le 0$ for all $\mu > 0$. Indeed, let $u \in C_0^\infty(R^N)$ and

$$\int_{\mathbf{R}^N} a(x)\left|\frac{u(x/\sigma)}{\sigma^{N/q}}\right|^p dx + \int_{\mathbf{R}^N} b(x)\left|\frac{u(x/\sigma)}{\sigma^{N/q}}\right|^q dx = \mu, \tag{2.30}$$

then for each $\sigma > 0$ we have

$$I_\mu^\infty \le \frac{1}{p\sigma^{p+(p/q-1)N}}\int_{\mathbf{R}^N}|\nabla u|^p dx + \frac{1}{q\sigma^q}\int_{\mathbf{R}^N}|\nabla u|^q dx - \sigma^N\int_{\mathbf{R}^N}\overline{F}\left(\sigma^{-N/q}u\right)dx \longrightarrow 0 \tag{2.31}$$

as $\sigma \to \infty$.

Lemma 2.6. *Suppose that $I_\lambda^\infty < 0$ for some $\lambda > 0$, then I_μ^∞/μ is nonincreasing on $(0,\infty)$ and $\lim_{\mu\to 0^+}(I_\mu^\infty/\mu) = 0$. Moreover there exists $\lambda^* \le \lambda$ such that*

$$\frac{I_\mu^\infty}{\mu} > \frac{I_\lambda^\infty}{\lambda} \quad \text{for } \mu \in (0,\lambda^*). \tag{2.32}$$

Proof. We observe that

$$\begin{aligned} &\inf \frac{I^\infty(u)}{\int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx} \\ &\quad = \inf \frac{I^\infty\left(u\left(x/\sigma^{1/N}\right)\right)}{\int_{\mathbf{R}^N} a\left(x/\sigma^{1/N}\right)\left|u\left(x/\sigma^{1/N}\right)\right|^p dx + b\left(x/\sigma^{1/N}\right)\left|u\left(x/\sigma^{1/N}\right)\right|^q dx}. \end{aligned} \tag{2.33}$$

So if $\int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx = k$ and $\int_{\mathbf{R}^N} a(x/\sigma^{1/N})|u(x/\sigma^{1/N})|^p dx + b(x/\sigma^{1/N})|u(x/\sigma^{1/N})|^q dx = k$ then $I^\infty(u(x)) = I^\infty(u(x/\sigma^{1/N})) = I_k^\infty$.

We have that if $\sigma > 0$ and $\alpha > 0$ with $\int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx = \alpha$, then

$$\int_{\mathbf{R}^N} a\left(\frac{x}{\sigma^{1/N}}\right)\left|u\left(\frac{x}{\sigma^{1/N}}\right)\right|^p dx + b\left(\frac{x}{\sigma^{1/N}}\right)\left|u\left(\frac{x}{\sigma^{1/N}}\right)\right|^q dx = \sigma\alpha, \qquad I^\infty\left(u\left(\frac{x}{\sigma^{1/N}}\right)\right) = I_{\sigma\alpha}^\infty. \tag{2.34}$$

Consequently, for all $\alpha_1 > 0$ and $\alpha_2 > 0$ we have

$$I_{\alpha_1}^{\infty} = \inf\left\{ \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{(N-p)/N} \int_{\mathbf{R}^N} |\nabla u|^p dx + \frac{1}{q}\left(\frac{\alpha_1}{\alpha_2}\right)^{(N-q)/N} \int_{\mathbf{R}^N} |\nabla u|^q dx - \frac{\alpha_1}{\alpha_2}\int_{\mathbf{R}^N} \overline{F}(u)dx; \right.$$
$$\left. \int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx = \alpha_2 \right\}. \tag{2.35}$$

If $0 < \alpha_1 < \alpha_2$, then for each $\varepsilon > 0$ there exists $u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$ with $\int_{\mathbf{R}^N} a(x)|u|^p + b(x)|u|^q dx = \alpha_2$ such that

$$\begin{aligned} I_{\alpha_1}^{\infty} + \varepsilon &> \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}\right)^{(N-p)/N} \int_{\mathbf{R}^N} |\nabla u|^p dx + \frac{1}{q}\left(\frac{\alpha_1}{\alpha_2}\right)^{(N-q)/N} \int_{\mathbf{R}^N} |\nabla u|^q dx - \frac{\alpha_1}{\alpha_2}\int_{\mathbf{R}^N} \overline{F}(u)dx \\ &\geq \frac{\alpha_1}{\alpha_2}\left(\frac{1}{p}\int_{\mathbf{R}^N} |\nabla u|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla u|^q dx - \int_{\mathbf{R}^N} \overline{F}(u)dx\right) \geq \frac{\alpha_1}{\alpha_2} I_{\alpha_2}^{\infty}. \end{aligned} \tag{2.36}$$

This inequality yields

$$\frac{I_{\alpha_1}^{\infty}}{\alpha_1} > \frac{I_{\alpha_2}^{\infty}}{\alpha_2}. \tag{2.37}$$

Since $I_{\mu}^{\infty} \leq 0$ for all $\mu > 0$, we see that

$$\lim_{\mu \to 0} \frac{I_{\mu}^{\infty}}{\mu} = c \leq 0. \tag{2.38}$$

We claim that $c = 0$. Indeed, it follows from (2.36) and from the estimate obtained in the Lemma 2.1 that for every $0 < \mu < \lambda$ there exists an $u_\mu \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$, with $\int_{\mathbf{R}^N} a(x)|u_\mu|^p + b(x)|u_\mu|^q dx = \lambda$ such that

$$\begin{aligned} I_{\mu}^{\infty} + \mu^2 &> \frac{1}{p}\left(\frac{\mu}{\lambda}\right)^{(N-p)/N} \int_{\mathbf{R}^N} |\nabla u_\mu|^p dx + \frac{1}{q}\left(\frac{\mu}{\lambda}\right)^{(N-q)/N} \int_{\mathbf{R}^N} |\nabla u_\mu|^q dx - \frac{\mu}{\lambda}\int_{\mathbf{R}^N} \overline{F}(u_\mu)dx \\ &\geq \frac{\mu}{\lambda}\left(\frac{1}{p}\int_{\mathbf{R}^N} |\nabla u_\mu|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla u_\mu|^q dx - \int_{\mathbf{R}^N} \overline{F}(u_\mu)dx\right) \\ &\geq \frac{\mu}{\lambda}\left(C_1(\lambda)\int_{\mathbf{R}^N} |\nabla u_\mu|^p dx + C_2(\lambda)\int_{\mathbf{R}^N} |\nabla u_\mu|^q dx - C_3(\lambda)\right), \end{aligned} \tag{2.39}$$

where $C_1(\lambda) > 0$, $C_2(\lambda) > 0$, and $C_3(\lambda) > 0$ are constants. Hence

$$\mu^2 \geq \frac{\mu}{\lambda}\left(C_1(\lambda)\int_{\mathbf{R}^N} |\nabla u_\mu|^p dx + C_2(\lambda)\int_{\mathbf{R}^N} |\nabla u_\mu|^q dx - C_3(\lambda)\right), \tag{2.40}$$

that is, $\int_{\mathbf{R}^N}|\nabla u_\mu|^p dx \leq C_4(\lambda)$, $\int_{\mathbf{R}^N}|\nabla u_\mu|^p dx \leq C_5(\lambda)$ for some constant $C_4(\lambda), C_5(\lambda) > 0$ independent of μ. We see that there exists $\varepsilon_0 > 0$ and a sequence $\mu_n \longrightarrow 0$ such that

$$\int_{\mathbf{R}^N}\left|\nabla u_{\mu n}\right|^p dx \geq \varepsilon_0, \qquad \int_{\mathbf{R}^N}\left|\nabla u_{\mu n}\right|^q dx \geq \varepsilon_0. \tag{2.41}$$

If $\int_{\mathbf{R}^N}|\nabla u_{\mu_n}|^p dx \geq \varepsilon_0$ then $\int_{\mathbf{R}^N}|\nabla u_{\mu_n}|^q dx \geq \eta \geq 0$.

Then, using the fact that $\int_{\mathbf{R}^N}\overline{F}(u_{\mu_n})dx \leq C$ for some constant $C > 0$, we get

$$\frac{I_{\mu_n}^\infty}{\mu_n} + \mu_n \geq \frac{1}{p}\lambda^{-(N-p)/N}{\mu_n}^{-p/N}\varepsilon_0 + \frac{1}{q}\lambda^{-(N-q)/N}{\mu_n}^{-q/N}\eta - \frac{C}{\lambda} \longrightarrow \infty \tag{2.42}$$

as $\mu_n \longrightarrow 0$ and this contradicts the fact that $\lim_{\mu\to 0}(I_\mu^\infty/\mu) = c \leq 0$. Therefore

$$\lim_{\mu\to 0}\int_{\mathbf{R}^N}\left|\nabla u_\mu\right|^p dx = 0, \qquad \lim_{\mu\to 0} I_\mu^\infty = 0, \tag{2.43}$$

when $\int_{\mathbf{R}^N}|\nabla u_{\mu_n}|^p dx \geq \varepsilon_0 > 0$ we can use the same method to obtain that $\lim_{\mu\to 0}\int_{\mathbf{R}^N}|\nabla u_\mu|^q dx = 0$.

So

$$\lim_{\mu\to 0}\int_{\mathbf{R}^N}\left|\nabla u_\mu\right|^p dx = \lim_{\mu\to 0}\int_{\mathbf{R}^N}\left|\nabla u_\mu\right|^q dx = 0, \qquad \lim_{\mu\to 0} I_\mu^\infty = 0, \tag{2.44}$$

this implies that $\lim_{\mu\to 0}\int_{\mathbf{R}^N}\overline{F}(u_\mu)dx = 0$ and consequently

$$\frac{I_\mu^\infty}{\mu} + \mu \geq -\frac{1}{\lambda}\int_{R^N}\overline{F}(u_\mu)dx \longrightarrow 0. \tag{2.45}$$

This shows that $\lim_{\mu\to 0^+}(I_\mu^\infty/\mu) = 0$. Finally, we observe that $\lim_{\mu\to 0}(I_\mu^\infty/\mu) = 0 > I_\lambda^\infty/\lambda$ which obtain (2.32). □

3. Proof of Main Theorems

Theorem 3.1. *Suppose that $I_\lambda^\infty < 0$ for some $\lambda > 0$, then there exists $0 < \alpha_0 \leq \lambda$ such that problem $(I_{\alpha_0}^\infty)$ has a minimizer. Moreover each minimizing sequence for $(I_{\alpha_0}^\infty)$ up to a translation is relatively compact in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$.*

Proof. According to Lemma 2.6 the set

$$\left\{\alpha_1;\ \frac{I_\alpha^\infty}{\alpha} > \frac{I_\lambda^\infty}{\lambda} \text{ for each } \alpha \in (0,\alpha_1)\right\} \tag{3.1}$$

is nonempty. We define

$$\alpha_0 = \sup\left\{\alpha_1;\ \frac{I_\alpha^\infty}{\alpha} > \frac{I_\lambda^\infty}{\lambda} \text{ for each } \alpha \in (0,\alpha_1)\right\}. \tag{3.2}$$

It follows from the continuity of I_λ^∞ that

$$\begin{gathered} 0 < \alpha_0 \leq \lambda\ , \\ I_{\alpha_0}^\infty = \frac{\alpha_0}{\lambda} I_\lambda^\infty < 0, \\ I_\alpha^\infty > \frac{\alpha}{\lambda} I_\lambda^\infty, \end{gathered} \tag{3.3}$$

for all $0 < \alpha < \alpha_0$. This yields

$$I_{\alpha_0}^\infty = \frac{\alpha_0}{\lambda} I_\lambda^\infty = \frac{\alpha_0 - \alpha}{\lambda} I_\lambda^\infty + \frac{\alpha}{\lambda} I_\lambda^\infty < I_{\alpha_0-\alpha}^\infty + I_\alpha^\infty \tag{3.4}$$

for each $\alpha \in (0,\alpha_0)$.

Let $\{u_m\} \subset W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$ be a minimizing sequence for $I_{\alpha_0}^\infty$. Since $\{u_m\}$ is bounded we may assume that $u_m \rightharpoonup u$ in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$, $u_m \to u$ a.e. on $\mathbf{R}^N$. First we consider the case $u \equiv 0$. In this case by Lemma 2.2 $u_m \to 0$ for $q < \alpha < p^*$ or Corollary there exists a sequence $\{u_m\} \subset \mathbf{R}^N$ such that $u_m(\cdot + y_m) \rightharpoonup v \neq 0$ in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$.

In the first case $\lim_{m\to\infty}\int_{\mathbf{R}^N} \overline{F}(u_m)dx = 0$ and consequently

$$I_{\alpha_0}^\infty = \lim_{m\to\infty} I^\infty(u_m) = \lim_{m\to\infty}\left(\frac{1}{p}\int_{\mathbf{R}^N}|\nabla u_m|^p dx + \frac{1}{q}\int_{\mathbf{R}^N}|\nabla u_m|^q dx - \int_{\mathbf{R}^N}\overline{F}(u_m)dx\right) \geq 0, \tag{3.5}$$

which is impossible. Hence $u_m(\cdot + y_m) \rightharpoonup v \neq 0$ in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$ holds and letting $v_m(x) = u_m(x + y_m)$ from Brezis-Lieb lemma (see [21, Lemma 1.32]) we have

$$\begin{aligned} \int_{\mathbf{R}^N} a(x)|u_m|^p + b(x)|u_m|^q dx &= \int_{\mathbf{R}^N} a(x+y_m)|v_m|^p + b(x+y_m)|v_m|^q dx \\ &= \int_{\mathbf{R}^N} a(x+y_m)|v|^p + b(x+y_m)|v|^q dx \\ &\quad + \int_{\mathbf{R}^N} a(x+y_m)|v_m - v|^p \\ &\quad + b(x+y_m)|v_m - v|^q dx + o(1). \end{aligned} \tag{3.6}$$

We now show that

$$\int_{\mathbf{R}^N} a(x+y_m)|v|^p + b(x+y_m)|v|^q dx = \alpha_0. \tag{3.7}$$

In the contrary case from Lemma 2.1 we have

$$0 < \int_{\mathbf{R}^N} a(x+y_m)|v|^p + b(x+y_m)|v|^q dx < \alpha_0. \tag{3.8}$$

By (3.21) we have

$$\begin{gathered}\lim_{m\to\infty}\int_{\mathbf{R}^N} a(x+y_m)|v_m - v|^p + b(x+y_m)|v_m - v|^q dx \longrightarrow \alpha_0 - \lambda, \\ \lambda = \int_{\mathbf{R}^N} a(x+y_m)|v|^p + b(x+y_m)|v|^q dx.\end{gathered} \tag{3.9}$$

On the other hand, by Lemmas 2.1 and 2.4 we have

$$\begin{gathered}\int_{\mathbf{R}^N} \overline{F}(v_m)dx = \int_{\mathbf{R}^N} \overline{F}(v)dx + \int_{\mathbf{R}^N} \overline{F}(v_m - v)dx + o(1), \\ \int_{\mathbf{R}^N} |\nabla v_m|^p + |\nabla v_m|^q dx \geq \int_{\mathbf{R}^N} |\nabla v|^p + |\nabla v|^q dx + \int_{\mathbf{R}^N} |\nabla(v_m - v)|^p + |\nabla(v_m - v)|^q dx + o(1),\end{gathered} \tag{3.10}$$

and this implies that

$$I_{\alpha_0}^{\infty} \geq I^{\infty}(v) + I^{\infty}(v_m - v) + o(1) \geq I_{\lambda}^{\infty} + I_{\alpha_0-\lambda_0}^{\infty} + o(1). \tag{3.11}$$

Letting $m \to \infty$ we get $I_{\alpha_0}^{\infty} \geq I_{\lambda}^{\infty} + I_{\alpha_0-\lambda_0}^{\infty}$ which contradicts (3.4). Therefore $\int_{\mathbf{R}^N} a(x+y_m)|v|^p + b(x+y_m)|v|^q dx = \alpha_0$. It then follows from (3.6) that $v_m \to v$ in $L^p \cap L^q(\mathbf{R}^N)$. By the Gagliardo-Nirenberg inequality $v_m \to v$ in $L^s(\mathbf{R}^N)$, $q \leq s < \infty$. Obviously this implies that $I_{\alpha_0}^{\infty} = I^{\infty}(v) = I^{\infty}(v(\cdot - y_m))$ and $\int_{\mathbf{R}^N} a(x)|v(\cdot - y_m)|^p + b(x)|v(\cdot - y_m)|^q dx = \alpha_0$. To complete the proof we show that $v_m \to v$ in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$. Indeed, we have

$$\begin{aligned}I_{\alpha_0}^{\infty} &= \frac{1}{p}\int_{\mathbf{R}^N} |\nabla v_m|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla v_m|^q dx - \int_{\mathbf{R}^N} \overline{F}(v_m)dx + o(1) \\ &\geq \frac{1}{p}\int_{\mathbf{R}^N} |\nabla v|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla v|^q dx + \frac{1}{p}\int_{\mathbf{R}^N} |\nabla v_m - v|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla v_m - v|^q dx \\ &\quad - \int_{\mathbf{R}^N} \overline{F}(v)dx + \int_{\mathbf{R}^N} \left(\overline{F}(v) - \overline{F}(v_m)\right)dx + o(1).\end{aligned} \tag{3.12}$$

Since $\lim_{m\to\infty}\int_{\mathbf{R}^N}(\overline{F}(v) - \overline{F}(v_m))dx = 0$, we deduce from (3.12) that $\nabla v_m \to \nabla v$ in $L^p \cap L^q(\mathbf{R}^N)$ and hence $v_m \to v$ in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$.

If $u \neq 0$, we repeat the previous argument to show that $I_{\alpha_0}^{\infty}$ is attained. □

Theorem 3.2. *Suppose that $F(x,t) \geq \overline{F}(t)$ on $\mathbf{R}^N \times \mathbf{R}$ and that $I_{\lambda} < 0$ for some $\lambda > 0$, then the infimum I_{λ_0} is attained for some $0 < \lambda_0 \leq \lambda$.*

Proof. Since $F(x,t) \geq \overline{F}(t)$ on $\mathbf{R}^N \times \mathbf{R}$ we have $I_\mu \leq I_\mu^\infty$ for $\mu \geq 0$. We distinguish two cases: (i) $I_\lambda = I_\lambda^\infty < 0$, (ii) $I_\lambda < I_\lambda^\infty$.

Case (i). By Theorem 3.1 there exists $\lambda_0 \in (0, \lambda]$ such that

$$I_{\lambda_0}^\infty = I^\infty(\overline{u}), \quad \int_{\mathbf{R}^N} a(x)|\overline{u}|^p + b(x)|\overline{u}|^q dx = \lambda_0 \quad \text{for some } \overline{u} \in W^{1,p} \cap W^{1,q}\left(\mathbf{R}^N\right). \tag{3.13}$$

Thus

$$\begin{aligned} I_{\lambda_0} \leq I(\overline{u}) &= \frac{1}{p}\int_{\mathbf{R}^N} |\nabla \overline{u}|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla \overline{u}|^q dx - \int_{\mathbf{R}^N} F(x, \overline{u}) dx \\ &\leq \frac{1}{p}\int_{\mathbf{R}^N} |\nabla \overline{u}|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla \overline{u}|^q dx - \int_{\mathbf{R}^N} \overline{F}(\overline{u}) dx = I^\infty(\overline{u}) = I_{\lambda_0}^\infty. \end{aligned} \tag{3.14}$$

If $I_{\lambda_0} = I_{\lambda_0}^\infty$, then I also attains its infimum I_{λ_0} at $\overline{u}$. Therefore it remains to consider the case $I_{\lambda_0} < I_{\lambda_0}^\infty$. Consequently we need to prove the following claim.

If $I_\lambda < I_\lambda^\infty$ for some $\lambda > 0$, then there exists $\alpha_0 \in (0, \lambda]$ such that problem (I_{α_0}) has a solution. This obviously completes the proof of case (i) and also provides the proof of case (ii).

By virtue of Lemma 2.5, $I_\beta + I_{\lambda-\beta}^\infty$ is continuous for $\beta \in [0, \lambda]$ and also $I_0 = I_0^\infty = 0$. If $I_\lambda < I_\lambda^\infty$ for some $\lambda > 0$, then there exists $\gamma > 0$ such that

$$I_\lambda < I_\beta + I_{\lambda-\beta}^\infty \tag{3.15}$$

for all $\beta \in [0, \gamma)$. Let

$$\alpha_0 = \sup\left\{\gamma;\ I_\lambda < I_\beta + I_{\lambda-\beta}^\infty,\ \text{for } 0 \leq \beta < \gamma\right\}. \tag{3.16}$$

Then we have

$$\begin{aligned} I_\lambda &= I_{\alpha_0} + I_{\lambda-\alpha_0}^\infty, \\ I_\lambda &< I_\alpha + I_{\lambda-\alpha}^\infty \end{aligned} \tag{3.17}$$

for $0 \leq \alpha < \alpha_0$. This implies that

$$I_{\alpha_0} + I_{\lambda-\alpha_0}^\infty = I_\lambda < I_\lambda^\infty \leq 0, \tag{3.18}$$

and hence

$$I_{\alpha_0} < I_\lambda^\infty - I_{\lambda-\alpha_0}^\infty \leq I_{\alpha_0}^\infty \leq 0, \tag{3.19}$$

we show that I_{α_0} is attained by a $\overline{u} \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$ and every minimizing sequence for I_{α_0} is relatively compact in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$. Let $\{u_m\}$ be a minimizing sequence for I_{α_0}. Since

$\{u_m\}$ is bounded, we may assume that $u_m \rightharpoonup \overline{u}$ in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$, $u_m \rightharpoonup \overline{u}$ a.e. on $\mathbf{R}^N$. Arguing indirectly we assume that $\overline{u} \equiv 0$ on $\mathbf{R}^N$. We see that

$$\lim_{m\to\infty}\int_{B(0,R)} |F(x,u_m)|dx = \lim_{m\to\infty}\int_{B(0,R)} \left|\overline{F}(u_m)\right|dx = 0 \tag{3.20}$$

for each $R > 0$. We now write

$$\begin{aligned} I(u_m) &= \frac{1}{p}\int_{\mathbf{R}^N} |\nabla \overline{u_m}|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla \overline{u_m}|^q dx - \int_{\mathbf{R}^N} F(x,\overline{u_m})dx \\ &= I^\infty(u_m) + \int_{\mathbf{R}^N} \left(\overline{F}(u_m) - F(x,u_m)\right)dx. \end{aligned} \tag{3.21}$$

We show that

$$\lim_{m\to\infty}\int_{\mathbf{R}^N} \left(\overline{F}(u_m) - F(x,u_m)\right)dx = 0. \tag{3.22}$$

Towards this end we write

$$\begin{aligned} &\int_{\mathbf{R}^N} \left|\overline{F}(u_m) - F(x,u_m)\right|dx \\ &\quad \le \int_{B(0,R)} |F(x,u_m)|dx + \int_{B(0,R)} \left|\overline{F}(u_m)\right|dx \\ &\qquad + \left(\int_{|x|\ge R, |u_m|\le\delta} + \int_{|x|\ge R, \delta\le|u_m|\le 1/\delta} + \int_{|x|\ge R, |u_m|>1/\delta}\right)\left|\overline{F}(u_m) - F(x,u_m)\right|. \end{aligned} \tag{3.23}$$

We now define the following quantities:

$$\begin{aligned} \epsilon_1(\delta) &= \sup_{0<|t|<\delta, x\in\mathbf{R}^N} \frac{\left|\overline{F}(t) - F(x,t)\right|}{|t|^p} \\ \epsilon(R) &= \sup_{\delta\le|t|\le 1/\delta, |x|\ge R} \left|\overline{F}(t) - F(x,t)\right|, \\ \epsilon_2(\delta) &= \sup_{|t|\ge 1/\delta} \frac{\left|\overline{F}(t) - F(x,t)\right|}{|t|^{pN/(N-p)}}. \end{aligned} \tag{3.24}$$

It follows from assumption (A) that $\lim_{\delta\to 0}\epsilon_1(\delta) = \lim_{\delta\to 0}\epsilon_2(\delta) = 0$ and by (B) $\lim_{R\to\infty}\epsilon(R) = 0$ for each fixed $\delta > 0$. Inserting these quantities into (3.23) we derive the following estimate:

$$\begin{aligned}
&\int_{\mathbf{R}^N}\left|\overline{F}(u_m) - F(x,u_m)\right|dx \\
&\quad\le \epsilon_1(\delta)\int_{\mathbf{R}^N}|u_m|^p dx + \frac{\epsilon(R)}{\delta^p}\int_{\mathbf{R}^N}|u_m|^p dx \\
&\qquad + \epsilon_2(\delta)\int_{\mathbf{R}^N}|u_m|^{pN/(N-p)}dx + \int_{B(0,R)}|F(x,u_m)|dx + \int_{B(0,R)}\left|\overline{F}(u_m)\right|dx.
\end{aligned}\tag{3.25}$$

First letting $m \to \infty$, $R \to \infty$, and then $\delta \to 0$, relation (3.22) readily follows. Combining (3.21) and (3.22) we obtain

$$I(u_m) = I^\infty(u_m) + o(1),\tag{3.26}$$

which implies $I(u_m) \ge I^\infty_{\alpha_0} + o(1)$ and consequently $I_{\alpha_0} \ge I^\infty_{\alpha_0}$ and this contradicts (3.19). Therefore $0 < \int_{\mathbf{R}^N} a(x)|\overline{u}|^p + b(x)|\overline{u}|^q dx \le \alpha_0$. Suppose that $\lambda = \int_{\mathbf{R}^N} a(x)|\overline{u}|^p + b(x)|\overline{u}|^q dx < \alpha_0$. Writing

$$\begin{aligned}
\int_{\mathbf{R}^N}|\nabla u_m|^p + |\nabla u_m|^q dx &\ge \int_{\mathbf{R}^N}|\nabla\overline{u}|^p + |\nabla\overline{u}|^q dx + \int_{\mathbf{R}^N}|\nabla(u_m-\overline{u})|^p + |\nabla(u_m-\overline{u})|^q dx + o(1) \\
\int_{\mathbf{R}^N} a(x)|u_m|^p + b(x)|u_m|^q dx &= \int_{\mathbf{R}^N} a(x)|\overline{u}|^p + b(x)|\overline{u}|^q dx \\
&\quad + \int_{\mathbf{R}^N} a(x)|u_m-\overline{u}|^p + b(x)|u_m-\overline{u}|^q dx + o(1),
\end{aligned}\tag{3.27}$$

$$\int_{\mathbf{R}^N}F(x,u_m)dx = \int_{\mathbf{R}^N}F(x,\overline{u})dx + \int_{\mathbf{R}^N}F(x,u_m-\overline{u})dx + o(1),\tag{3.28}$$

we deduce that

$$I_{\alpha_0} \ge I(\overline{u}) + I(u_m-\overline{u}) + o(1).\tag{3.29}$$

By a similar method used to obtain (3.22) we also have

$$I(u_m-\overline{u}) = I^\infty(u_m-\overline{u}) + o(1).\tag{3.30}$$

Hence the last two relations yield

$$I_{\alpha_0} \ge I(\overline{u}) + I^\infty(u_m-\overline{u}) + o(1) \ge I_\lambda + I^\infty_{\alpha_0-\lambda},\tag{3.31}$$

and this contradicts (3.19). Consequently $\int_{\mathbf{R}^N} a(x)|\overline{u}|^p + b(x)|\overline{u}|^q dx = \alpha_0$ and (3.27) yields $u_m \to \overline{u}$ in $L^p \cap L^q(\mathbf{R}^N)$. By the Gagliardo-Nirenberg inequality we have $u_m \to \overline{u}$ in $L^s(\mathbf{R}^N)$, $q \le s < p^*$. This obviously show that $I_{\alpha_0} = I(\overline{u})$ and $\int_{\mathbf{R}^N} a(x)|\overline{u}|^p + b(x)|\overline{u}|^q dx = \alpha_0$; that is, $\overline{u}$ is a solution of problem (I_{α_0}). Finally, writing

$$\begin{aligned} I_{\alpha_0} \ge{}& \frac{1}{p}\int_{\mathbf{R}^N} |\nabla\overline{u}|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla\overline{u}|^q dx + \frac{1}{p}\int_{\mathbf{R}^N} |\nabla(u_n - \overline{u})|^p dx + \frac{1}{q}\int_{\mathbf{R}^N} |\nabla(u_n - \overline{u})|^q dx \\ &- \int_{\mathbf{R}^N} \overline{F}(\overline{u}) dx - \int_{\mathbf{R}^N} \left(F(x,u_m) - \overline{F}(u_m)\right) dx + \int_{\mathbf{R}^N} \left(\overline{F}(\overline{u}) - \overline{F}(u_m)\right) dx + o(1), \end{aligned} \tag{3.32}$$

and using (3.22) we deduce from this that $\nabla u_m \to \nabla\overline{u}$ in $L^p \cap L^q(\mathbf{R}^N)$ and hence $u_m \to \overline{u}$ in $W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$.

Theorem 3.3. *Suppose that $F(x,t) \ge \overline{F}(t)$ on $\mathbf{R}^N \times \mathbf{R}$ and that $\overline{F}(\zeta) > 0$ for some $\zeta \in \mathbf{R}$, then problem (I_λ) has a minimizer for some $\lambda > 0$.*

Proof. The condition $\overline{F}(\zeta) > 0$ for some $\zeta > 0$ implies that $\int_{\mathbf{R}^N} \overline{F}(u(x)) dx > 0$ for some $u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N)$. Letting $v(x) = u(x/\sigma)$, $\sigma > 0$, we have

$$I^\infty(v) = \sigma^N \left(\frac{1}{p\sigma^p}\int_{\mathbf{R}^N} |\nabla u|^p dx + \frac{1}{q\sigma^q}\int_{\mathbf{R}^N} |\nabla u|^q dx - \int_{\mathbf{R}^N} \overline{F}(u(x)) dx \right) < 0 \tag{3.33}$$

for $\sigma > 0$ sufficiently large. Hence there exists $\lambda > 0$ such that $I_\lambda \le I_\lambda^\infty < 0$ and the result follows from Theorem 3.2. □

Remark 3.4. It is a standard argument that minimizers of I_μ correspond to weak solutions of problem (1.1) with λ appearing as a Lagrange multiplier. Such a λ is then called the principal eigenvalue for problem (1.1).

Remark 3.5. If $a \in L^{N/p}(\mathbf{R}^N)$, $b \in L^{N/q}(\mathbf{R}^N)$, $a, b < 0$, we can use the similar method to study it, where $I_\lambda = \inf\{I(u); u \in W^{1,p} \cap W^{1,q}(\mathbf{R}^N), \int_{\mathbf{R}^N} a^-(x)|u|^p + b^-(x)|u|^q dx = \lambda\}, \lambda > 0, a^- = -a,\ b^- = -b$. □

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